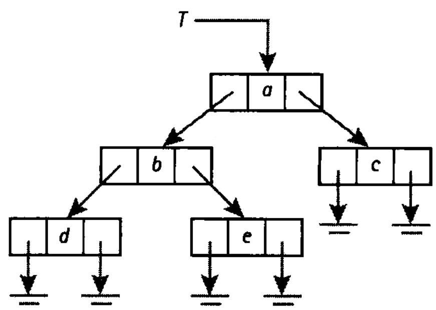


Figure 1.27 Computer representation of a binary tree.

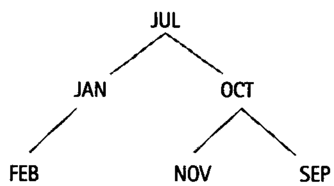


Figure 1.28 Binary search tree.

of the tree, each element in its left subtree precedes the node element and each element in its right subtree succeeds the node element.

example 1.43 A Binary Search Tree

The binary search tree in Figure 1.28 holds three-letter abbreviations for six of the months, where we are using the dictionary ordering of the words. So the correct order is FEB, JAN, JUL, NOV, OCT, SEP.

This binary search tree has depth 2. There are many other binary search trees to hold these six months. Find another one that has depth 2. Then find one that has depth 3.

end example

1.4.5 Spanning Trees

A *spanning tree* for a connected graph is a subgraph that is a tree and contains all the vertices of the graph. For example, Figure 1.29 shows a graph followed by two of its spanning trees. This example shows that a graph can have many spanning trees. A *minimal spanning tree* for a connected weighted graph is a spanning tree such that the sum of the edge weights is minimum among all spanning trees.

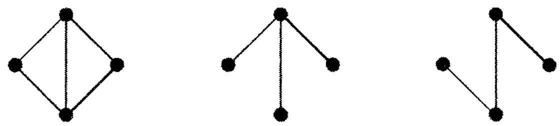


Figure 1.29 Two spanning trees.

Prim's Algorithm

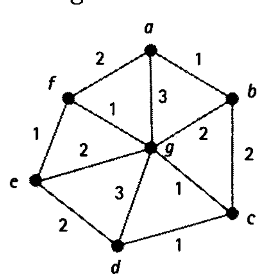
A famous algorithm, due to Prim [1957], constructs a minimal spanning tree for any undirected connected weighted graph. Starting with any vertex, the algorithm searches for an edge of minimum weight connected to the vertex. It adds the edge to the tree and then continues by trying to find new edges of minimum weight such that one vertex is in the tree and the other vertex is not. Here's an informal description of the algorithm.

> **Prim's Algorithm**
>
> Construct a minimal spanning tree for an undirected connected weighted graph. The variables: V is the vertex set of the graph; W is the vertex set and S is the edge set of the spanning tree.
>
> **1.** Initialize $S := \varnothing$.
>
> **2.** Pick any vertex $v \in V$ and set $W := \{v\}$.
>
> **3. while** $W \neq V$ **do**
>
> Find a minimum weight edge $\{x, y\}$, where $x \in W$ and $y \in V - W$;
>
> $S := S \cup \{\{x, y\}\}$;
>
> $W := W \cup \{y\}$
>
> **od**

Of course, Prim's algorithm can also be used to find a spanning tree for an unweighted graph. Just assign a weight of 1 to each edge of the graph. Or modify the first statement in the **while** loop to read "Find an edge $\{x, y\}$ such that $x \in W$ and $y \in V - W$."

example 1.44 A Minimal Spanning Tree

We'll construct a minimal spanning tree for the following weighted graph.



To see how the algorithm works we'll do a trace of each step showing the values of the variables S and W. The algorithm gives us several choices since it is not implemented as a computer program. So we'll start with the letter a since it's the first letter of the alphabet.

S	W
$\{\}$	$\{a\}$
$\{\{a,b\}\}$	$\{a,b\}$
$\{\{a,b\},\{b,c\}\}$	$\{a,b,c\}$
$\{\{a,b\},\{b,c\},\{c,d\}\}$	$\{a,b,c,d\}$
$\{\{a,b\},\{b,c\},\{c,d\},\{c,g\}\}$	$\{a,b,c,d,g\}$
$\{\{a,b\},\{b,c\},\{c,d\},\{c,g\},\{g,f\}\}$	$\{a,b,c,d,g,f\}$
$\{\{a,b\},\{b,c\},\{c,d\},\{c,g\},\{g,f\},\{f,e\}\}$	$\{a,b,c,d,g,f,e\}$

The algorithm stops because $W = V$. So S is a spanning tree.

end example

Exercises

Graphs

1. Draw a picture of a graph that represents those states of the United States and those provinces of Canada that touch the Atlantic Ocean or border states or provinces that do.

2. Find planar graphs with the smallest possible number of vertices that have chromatic numbers of 1, 2, 3, and 4.

3. What is the chromatic number of the graph representing the map of the United States? Explain your answer.

4. Draw a picture of the directed graph that corresponds to each of the following binary relations.

   a. $\{(a, a), (b, b), (c, c)\}$.

   b. $\{(a, b), (b, b), (b, c), (c, a)\}$.

   c. The relation $\leq$ on the set $\{1, 2, 3\}$.

5. Given the following graph: