Assignment 1

Your name here

August 21, 2024

1. One of the constructions in this class will take a set of states Q for one machine and create a new machine which uses a set of states R which is the set of all subsets of Q (you don't need to know what a "machine" or "state" really is in this – it's all about the sets).

   (a) What is the mathematical terminology? In other words, R is the ______________ of Q.

   (b) If $Q = \{a, b, c\}$, what is R?

   (c) Prove that for any set Q, the size of R satisfies $|R| = 2^{|Q|}$.

2. Describe each of the following sets in high-level plain English (no formulas!). For example, the set $\{x \mid x = 2k + 1 \text{ for } k \geq 1\}$ is "The set of odd integers greater than or equal to 3." Do **not** simply restate things as written here (in other words, do not say "The set of integers x for which $x = 2k + 1$ and $k \geq 1$").

   The sets below are all sets of binary strings (note that $\{0,1\}^*$ denotes the set of all binary strings), and in addition to the notation in the book (pages 13–14) we define $c_0(x)$ to be the number of 0's in string x, and $c_1(x)$ to be the number of 1's in x. For example, if $x = \texttt{011010001}$ then $c_0(x) = 5$ and $c_1(x) = 4$.

   (a) $\{x \mid x \in \{0,1\}^* \text{ and } |x| = 2k \text{ for some integer } k\}$

   (b) $\{x \mid x \in \{0,1\}^* \text{ and } c_0(x) = c_1(x)\}$

   (c) $\{x \mid x \in \{0,1\}^* \text{ and } c_1(x) = 2k + 1 \text{ for some integer } k\}$

   (d) $\{x \mid x \in \{0,1\}^* \text{ and } x = x^{\mathcal{R}}\}$

3. We define a function f to update the position of a robot on a 3-position number-line, with valid positions -1, 0, and $+1$, and an "out of bounds" position "out". The set of positions is denoted $P = \{\text{out}, -1, 0, +1\}$. The valid set of moves is $M = \{\text{Left}, \text{Right}\}$. Once out of bounds, the robot cannot come back to a valid position. The position update function $f : P \times M \to P$ is given in the following table:

	Left	Right
out	out	out
-1	out	0
0	-1	$+1$
$+1$	0	out

   For example, $f(0, \text{Right}) = +1$ and $f(-1, \text{Left}) = \text{out}$.

(a) Explain in your own words what the notation "$f : P \times M \to P$" means.

(b) What is $f(f(f(f(f(0, \text{Left}), \text{Right}), \text{Right}), \text{Left}), \text{Right})$? **Show your work!**

(c) What is $f(f(f(f(f(0, \text{Left}), \text{Right}), \text{Right}), \text{Right}), \text{Left})$? **Show your work!**

4. Prove by contradiction: Any undirected graph with $n \geq 2$ vertices must have two vertices with the same degree.

5. Use induction to prove that for all $n \geq 1$,

$$\sum_{x=0}^{n-1} x(x-1) = \frac{n(n-1)(n-2)}{3}.$$

6. Let G be a directed graph with n vertices, and let v and w be any two nodes in the graph. Prove that if there is a path from node v to node w of length n, then there is another path from u to v with length greater than $2n$. (*Hint: Think about what is appropriate in the following blank, and how that is important to this property: "Any path with n edges in an n-vertex graph must contain a ______________." Note that this is just to get you thinking. Any property you fill in that blank with needs to be proved as part of the overall proof – you can't just state it, even if you think it's obvious.*)