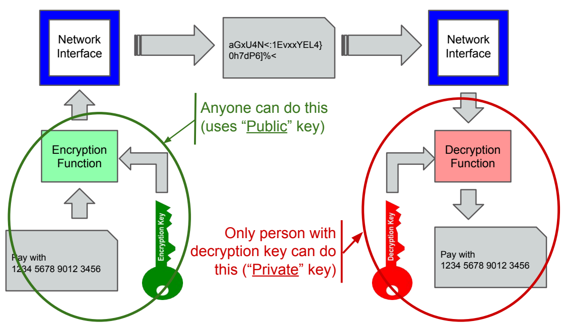
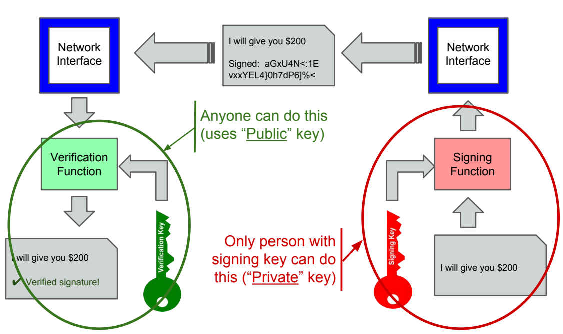
CSC 580
Cryptography and Computer Security

Digital Signatures
(Sections 13.1, 13.2, 13.4, 13.6)

Digital Signatures - Idea
Public key encryption idea



Digital Signatures - Idea
Digital signature idea



Digital Signatures - How it Works

Signature scheme consists of three algorithms:

- *Generate keypair*: Given keylength (security param) gives (PU,PR)
- *Sign*: Takes message M and PR, and produces signature sig
- *Verify*: Takes M, PU, and sig, and outputs true (verified) or false

Like public key encryption, sign/verify operations are slow!

- So don't run entire (possibly long) message through functions
- First hash, then sign H(M)

Is this combination secure? Yes! Why: Assume adversary knows valid sigs (M_1,sig_1), (M_2,sig_2), …, (M_n,sig_n) and can find a forgery (M,sig).

- If $H(M) = H(M_i)$ for some M_i $\rightarrow$ found a collision in H, should be impossible!
- If $H(M) \neq H(M_i)$ for all M_i $\rightarrow$ then $(H(M),sig)$ is a forger for sig scheme

Digital Signatures - Security Model

```
A^S(PU)
   // Arbitrary precomputation
   while (not done):
      m = // compute query message
      s = S(m)
      Known = Known ∪ (m,s)
      // More computing
   (m', s') = // compute claimed forgery
   Return (m',s')
```

Adversary wins if there is no pair `(m',x)` in Known and Verify`(m',s')` = true

Note:

- Adversary picks oracle query messages, and can adapt as it learns
  - That makes this an "adaptive chosen message" attack
- Any valid signature wins - only restriction is that m' hasn't been queried
  - That makes this an "existential forgery attack"

Security is Existentially Unforgeable under Adaptive Chosen Message Attack (EUF-CMA)

ElGamal

As in Diffie-Hellman, let p be a prime and g be a primitive root

Key Generation

Note similarity to Diffie-Hellman

1. Pick random $PR \in \{2, \ldots, p\text{-}1\}$
2. Compute $PU = g^{PR} \bmod p$
3. Private (signing) key is PR ; Public (verification) key is PU

Signing a message M

1. Pick random $k \in \{2, \ldots, p\text{-}1\}$ that is relative prime to (p-1)
2. Compute $r = g^k \bmod p$
3. Compute $k^{-1} \bmod (p\text{-}1)$
4. Compute $s = k^{-1} (H(M) - PR^*r) \bmod (p\text{-}1)$
5. Signature is the pair (r,s)

Verifying a signature (r,s) on message M:

1. Check if $g^{H(M)} \equiv PU^r * r^s \pmod p$ [accept if true, reject if false]

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5. Signature is the pair (r,s)

Observation: Expensive computations (powering and inverse) don't depend on M - precompute!

Verifying a signature (r,s) on message M:

1. Check if $g^{\mathrm{H}(M)} \equiv PU^{r} * r^{s} \pmod p$ [accept if true, reject if false]

Why does this work for valid sigs?

Important math fact: If $x \equiv y \pmod{p\text{-}1}$ then $a^x \equiv a^y \pmod p$.

Proof: If $x \equiv y \pmod{p\text{-}1}$ then there exists a k such that $x\text{-}y = k^*(p\text{-}1)$, so $x = k^*(p\text{-}1)+y$. Then $a^x = a^{k^*(p-1)+y} = a^{k^*(p-1)}{}^*a^y = (a^{p-1})^{k}{}^*a^y$. By Fermat's Little Theorem, we know that $a^{p-1} \bmod p = 1$, so $(a^{p-1})^{k}{}^*a^y \bmod p = a^y$. Therefore $a^x \equiv a^y \pmod p$.

What this means: To simplify $a^{formula}$, can simplify *formula* mod $(p\text{-}1)$.

Applying this to ElGamal formulas:

$PU = g^{PR} \bmod p$

$s = k^{-1}\,(\mathrm{H}(M) - PR^*r) \bmod (p\text{-}1)$

Consider $PU^{r} * r^{s} \equiv g^{PR^*r}\, g^{k^*s} \equiv g^{PR^*r+k^*s}$ (mod p), and simplify exponent mod (p-1):

$PR^*r + k^*s \equiv PR^*r + k^*k^{-1}\,(\mathrm{H}(M) - PR^*r) \equiv PR^*r + \mathrm{H}(M) - PR^*r \equiv \mathrm{H}(M) \bmod (p\text{-}1)$

Therefore, $PU^{r} * r^{s} \equiv g^{\mathrm{H}(M)} \pmod p$

DSA - Digital Signature Algorithm

Compared to ElGamal

ElGamal

Let $q = p\text{-}1$

Key Generation

1. Pick random $PR \in \{2, \ldots, q\}$
2. Compute $PU = g^{PR} \bmod p$
3. Private key is PR ; Public key is PU

Signing a message M

1. Pick rand $k \in \{2, \ldots, q\}$ with $\gcd(k,q)=1$
2. Compute $r = g^k \bmod p$
3. Compute $k^{-1} \bmod q$
4. Compute $s = k^{-1}\,(\mathrm{H}(M) - PR^*r) \bmod q$
5. Signature is the pair (r,s)

Verifying signature (r,s) on message M:

1. Check if $g^{\mathrm{H}(M)} \equiv PU^{r} * r^{s} \pmod p$

DSA

q is prime such that $q|p\text{-}1$, and let g be a value with order q [$g^q \equiv 1 \pmod q$]

Key Generation

1. Pick random $PR \in \{2, \ldots, q\}$
2. Compute $PU = g^{PR} \bmod p$
3. Private key is PR ; Public key is PU

Signing a message M

1. Pick rand $k \in \{2, \ldots, q\text{-}1\}$
2. Compute $r = (g^k \bmod p) \bmod q$
3. Compute $k^{-1} \bmod q$
4. Compute $s = k^{-1}\,(\mathrm{H}(M) + PR^*r) \bmod q$
5. Signature is the pair (r,s)

Verifying signature (r,s) on message M:

1. Compute $w = s^{-1} \bmod q$
2. Check if $r \equiv (PU^{r^*w} * g^{\mathrm{H}(M)^*w} \bmod p) \bmod q$

DSA - The Digital Signature Algorithm
History, Parameters, etc.

One component of NIST's Digital Signature Standard (DSS)

- DSS was adopted in 1993
- DSA dates back to 1991
- One goal: Only support integrity - not confidentiality
  - Why? Export restrictions!
  - Alternative signature scheme: RSA - also an encryption algorithm

Key and Parameter Sizes:

- ElGamal is similar to Diffie-Hellman modulus size (N = number of bits)
  - 1024-bit p was OK in 1990s - now suggest 2048-bit or 3072-bit
  - Signature two N-bit values (e.g., two 1024-bit values)
- DSA uses a computationally-hard subgroup
  - In 1990's q was 160 bits (matching SHA1!)
  - Signature was then two 160-bit values (more compact than ElGamal)
  - Now suggest q being 256 bits

Reminder - RSA Algorithm
From Public Key Encryption chapter

Key Generation:

Pick two large primes p and q
Calculate $n=p*q$ and $\phi(n)=(p-1)*(q-1)$
Pick a random e such that $\gcd(e, \phi(n))$
Compute $d = e^{-1} \pmod{\phi(n)}$ *[Use extended GCD algorithm!]*
Public key is $PU=(n,e)$; Private key is $PR=(n,d)$

Encryption of message $M \in \{0,..,n-1\}$:
$E(PU,M) = M^e \bmod n$

Decryption of ciphertext $C \in \{0,..,n-1\}$:
$D(PR,C) = C^d \bmod n$

Correctness - easy when $\gcd(M,n)=1$:

$D(PR,E(PU,M)) = (M^e)^d \bmod n$
$= M^{ed} \bmod n$
$= M^{k\phi(n)+1} \bmod n$
$= (M^{\phi(n)})^k M \bmod n$
$= M$

Also works when $\gcd(M,n)\neq1$, but slightly harder to show...

RSA Algorithm for Signatures
"Textbook algorithm" - not how it's really done

Key Generation:

Pick two large primes p and q
Calculate $n=p*q$ and $\phi(n)=(p-1)*(q-1)$
Pick a random v such that $\gcd(v, \phi(n))$
Compute $s = v^{-1} \pmod{\phi(n)}$ *[Use extended GCD algorithm!]*
Public key is $PU=(n,v)$; Private key is $PR=(n,s)$

Signing message $M \in \{0,..,n-1\}$:
$\mathrm{Sign}(PR,M) = M^s \bmod n$

Verification of signature $\sigma \in \{0,..,n-1\}$:
$\mathrm{Verify}(PU,M,\sigma)$: Check if $M = \sigma^v \bmod n$

RSA-PSS (Probabilistic Signature Scheme)

How it's really done - with padding (similar to OAEP for encryption)

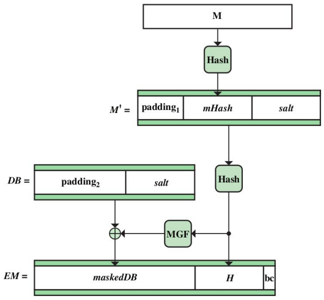

Figure 13.6 RSA-PSS Encoding

Invented (and proved secure) by Bellare and Rogaway

- Also inventors of OAEP and HMAC

Forging sigs w/ "textbook RSA"

- Pick random sig R
- Let message $M=R^v \bmod N$
- (M,R) is valid sig pair!

Modifying sigs ("blinding")

- Given $\sigma = M^s \bmod N$
- Compute $X = R^v \bmod N$
- Let $M' = X*M \bmod N$
- Let $\sigma' = R*\sigma \bmod N$
- Note $(\sigma')^v = R^v\sigma'^v = X*M = M'$ (mod N)