Assignment 1 – Problem listing (due Wednesday, September 11)

Remember to provide full solutions. Proofs should be complete, and computation/analysis problems should always show work or justification (never just state the final answer!).

1. Write a direct proof of the following theorem.

   Theorem 1. *If x and y are rational, then $x + y$ is also rational.*

2. Prove the following theorem using the contrapositive of the statement.

   Theorem 2. *If n is an integer such that $7n + 6$ is odd, then n is odd.*

3. Prove the following theorem using cases (this theorem is called the "Triangle Inequality").

   Theorem 3. *For all real numbers x and y, $|x + y| \leq |x| + |y|$.*

4. Prove the following theorem using contradiction.

   Theorem 4. *If $a, b \in \mathbb{R}$ such that a is rational and ab is irrational, then b is irrational.*

5. Convert "$((P$ OR NOT$(S))$ IMPLIES $(Q$ AND $R))$ OR S" into an equivalent proposition in DNF (using just AND, OR, and NOT).

6. Prove that "NOT$(P$ OR (NOT(P) AND $Q))$" and "NOT(P) AND NOT(Q)" are equivalent two ways:

   (a) Prove the equivalence using truth tables.

   (b) Prove the equivalence *without* truth tables, using just Boolean formula manipulation rules (distributive laws, De Morgan's laws, etc.)

7. Let $V(u, w)$ denote the predicate "User u has visited website w." Write the following English statements as quantified propositions.

   (a) Every user has visited some web site.

   (b) Every website has been visited by some user.

   (c) All users have visited `www.google.com`.

8. Let $F(0), F(1), F(2), \ldots$ denote the Fibonacci sequence, as in the textbook (see page 36). Prove the following theorem using induction.

   Theorem 5. *For all $n \geq 0$, $F(0) + F(1) + \cdots + F(n) = F(n + 2) - 1$.*

9. Prove the following theorem about "making change" using induction.

   Theorem 6. *If $n \geq 12$ is an integer, then n cents can be made using just 3 and 7 cent coins.*

10. Consider the following recursively-defined function.

```
function MyFunction(x, n)
    if n = 0 then
        return 1
    else
        return x * MyFunction(x, n − 1)
    end if
end function
```

Prove the following theorem using induction.

Theorem 7. *For all* $n \geq 0$, $\textsc{MyFunction}(x, n)$ *returns* x^n.

11. This question deals with "finite calculus," giving formulas for certain sums that should look similar to integral formulas from regular (infinite) calculus.

Definition 8. *For* $k \geq 1$*, define the* k*th "falling factorial power" of* x *by*

$$x^{\underline{k}} = \overbrace{x(x-1)\cdots(x-k+1)}^{k\text{ factors}}.$$

Prove the following theorem using induction.

Theorem 9. *For all integers* $n \geq 1$ *and* $k \geq 1$,

$$\sum_{x=0}^{n-1} x^{\underline{k}} = \frac{n^{\underline{k+1}}}{k+1}.$$